ON THE CONSTRUCTION OF PERIODIC SOLUTIONS OF A NONAUTONOMOUS QUASI-LINEAR SYSTEM WITH TWO DEGREES OF FREEDOM

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PMM Vol.24, No.5, 1960, pp. 933-937

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(Received 15 June 1960)

1. Let us consider a nonautonomous oscillatory system, with two degrees of freedom, of the form

$$\frac{d^2x}{dt^2} + ax + by = f(t) + \mu F, \qquad \frac{d^2y}{dt^2} + cx + dy = \varphi(t) + \mu\Phi \tag{1.1}$$

Let us suppose that the functions f and ϕ are continuous periodic functions of time t of period 2π. The functions F and Φ are assumed to be analytic in the variables x, x', y, y', μ, and continuous periodic functions of time t with the same period 2π. The coefficients a, b, c and d are constants. The quantity μ is a small parameter. In the case of resonance, the fundamental equation

$$\begin{vmatrix} D^2 + a & b \\ c & D^2 + d \end{vmatrix} = 0$$

will have either a zero root or roots of the form $\pm\, pi$ where p is an integer. Another instance, which will also be referred to as a resonance case, occurs when the fundamental equation has roots which differ from the indicated critical values by a quantity which is an infinitesimal of the order of μ. This case can be reduced, however, to the former one by the introduction of correcting terms into the functions μF and $\mu\Phi$.

Furthermore, we shall assume that the coefficients of the pth harmonic in the Fourier expansions of the functions $f(t)$ and $\phi(t)$ are either absent or are infinitesimals of the order of μ. The generating system $(\mu = 0)$

$$\frac{d^2x}{dt^2} + ax + by = f(t), \qquad \frac{d^2y}{dt^2} + cx + dy = \varphi(t) \tag{1.2}$$

can have the following families of periodic solutions:

a) If the fundamental equation has four roots $\pm ik$, $\pm im$, where k and m are integers and $k \neq m$, then

$$x_0(t) = x_0^{(k)}(t) + x_0^{(m)}(t) + f^\circ(t), \quad y_0(t) = p_k x_0^{(k)}(t) + p_m x_0^{(m)}(t) + \varphi^\circ(t) \qquad (1.3)$$

Here, $f^\circ(t)$ and $\phi^\circ(t)$ are the particular periodic solutions of (1.2) given by

$$x_0^{(k)}(t) = A_0 \cos kt + \frac{B_0}{k} \sin kt, \quad p_k = \frac{c}{k^2 - d} = \frac{k^2 - a}{b}$$
$$x_0^{(m)}(t) = E_0 \cos mt + \frac{D_0}{m} \sin mt, \quad p_m = \frac{c}{m^2 - d} = \frac{m^2 - a}{b}.$$

where A_0, B_0, E_0 and D_0 are arbitrary constants.

b) If the fundamental equation has only two roots $\pm ik$, where k is an integer, then

$$x_0(t) = x_0^{(k)}(t) + f^\circ(t), \qquad y_0(t) = p_k x_0^{(k)}(t) + \varphi^\circ(t) \qquad (1.4)$$

c) If the fundamental equation has multiple roots $k = m$, then

$$x_0(t) = A_0 \cos kt + \frac{B_0}{k} \sin kt + f^\circ(t), \qquad y_0(t) = E_0 \cos kt + \frac{D_0}{k} \sin kt + \varphi^\circ(t) \quad (1.5)$$

The case of zero roots is not considered in this work.

We shall look for periodic solutions of the fundamental system (1.1) by the method of a small parameter. Let us consider the case (a). The initial conditions are taken as

$$\begin{aligned} x(0) &= f^\circ(0) + A_0 + E_0 + \beta_1 + \beta_3 \\ x'(0) &= f^{\circ\prime}(0) + B_0 + D_0 + \beta_2 + \beta_4 \\ y(0) &= \varphi^\circ(0) + p_k A_0 + p_m E_0 + p_k \beta_1 + p_m \beta_3 \\ y'(0) &= \varphi^{\circ\prime}(0) + p_k B_0 + p_m D_0 + p_k \beta_2 + p_m \beta_4 \end{aligned} \qquad (1.6)$$

where the quantities β_i are functions of μ which vanish when $\mu = 0$. In this case the solution of the system (1.1) has the form

$$x = x(t, \beta_1, \beta_2, \beta_3, \beta_4, \mu), \qquad y = y(t, \beta_1, \beta_2, \beta_3, \beta_4, \mu$$

We shall determine the structure of these functions. Let us suppose that they can be expanded into series of integer powers of the parameters β_1, β_2, β_3, β_4 and μ. Let us find those terms of these series which depend on β_1, β_2, β_3 and β_4 but not on μ. It is easily seen that all these terms, with the exception of those that are linear in β_1, β_2, β_3 and β_4, vanish because their coefficients satisfy a system of homogeneous differential equations with zero initial conditions. The coefficients of the linear terms, obviously, have a form analogous to that part of the generating solution which corresponds to the periodic solution of the homogeneous equations corresponding to (1.2); one needs only to replace A_0, B_0, E_0 and D_0 by β_1, β_2, β_3 and β_4, respectively, in accordance with the form of the initial conditions (1.6).

Thus, the solution of Equation (1.1) can be given in the form [1]

$$x(t) = f^\circ(t) + (A_0 + \beta_1)\cos kt + \frac{B_0 + \beta_2}{k}\sin kt + (E_0 + \beta_3)\cos mt + \frac{D_0 + \beta_4}{m}\sin mt + \sum_{n=1}^{\infty}\left[C_n + \frac{\partial C_n}{\partial \beta_1}\beta_1 + \frac{\partial C_n}{\partial \beta_2}\beta_2 + \frac{\partial C_n}{\partial \beta_3}\beta_3 + \frac{\partial C_n}{\partial \beta_4}\beta_4 + \ldots\right]\mu^n \tag{1.7}$$

$$y(t) = \varphi^\circ(t) + p_k\left[(A_0 + \beta_1)\cos kt + \frac{B_0 + \beta_2}{k}\sin kt\right] + p_m\left[(E_0 + \beta_3)\cos mt + \frac{D_0 + \beta_4}{m}\sin mt\right] + \sum_{n=1}^{\infty}\left[H_n + \frac{\partial H_n}{\partial \beta_1}\beta_1 + \frac{\partial H_n}{\partial \beta_2}\beta_2 + \frac{\partial H_n}{\partial \beta_3}\beta_3 + \frac{\partial H_n}{\partial \beta_4}\beta_4 + \ldots\right]\mu^n$$

All the C_n, H_n and their derivatives with respect to β_i are taken with $\beta_1 = \ldots = \beta_4 = \mu = 0$. The coefficients C_n and H_n satisfy the system

$$C_n'' + aC_n + bH_n = F_n, \qquad H_n'' + cC_n + dH_n = \Phi_n \tag{1.8}$$

with the initial conditions $C_n(0) = H_n(0) = C_n'(0) = H_n'(0) = 0$. Here

$$F_n(t) = \frac{1}{(n-1)!}\left(\frac{d^{n-1}F}{d\mu^{n-1}}\right)_{\beta_i = \mu = 0}, \qquad \Phi_n = \frac{1}{(n-1)!}\left(\frac{d^{n-1}\Phi}{d\mu^{n-1}}\right)_{\beta_i = \mu = 0}$$

The quantities $d^{n-1}F/d^{n-1}$ and $d^{n-1}\phi/d^{n-1}$ are total derivatives of the functions $F(t, x, x', y, y', \mu)$ and $\phi(t, x, x', y, y', \mu)$ with respect to μ. We present the first three functions F_n in explicit forms

$$F_1(t) = F(t, x_0, x_0', y_0, y_0', 0)$$

$$F_2(t) = \left(\frac{\partial F}{\partial x}\right)_0 C_1 + \left(\frac{\partial F}{\partial x'}\right)_0 C_1' + \left(\frac{\partial F}{\partial y}\right)_0 H_1 + \left(\frac{\partial F}{\partial y'}\right)_0 H_1' + \left(\frac{\partial F}{\partial \mu}\right)_0$$

$$F_3(t) = \frac{1}{2}\left(\frac{\partial^2 F}{\partial x^2}\right)_0 C_1^2 + \frac{1}{2}\left(\frac{\partial^2 F}{\partial x'^2}\right)_0 C_1'^2 + \frac{1}{2}\left(\frac{\partial^2 F}{\partial y^2}\right)_0 H_1^2 + \frac{1}{2}\left(\frac{\partial^2 F}{\partial y'^2}\right)_0 H_1'^2 + \frac{1}{2}\left(\frac{\partial^2 F}{\partial \mu^2}\right)_0 + \left(\frac{\partial^2 F}{\partial x \partial y}\right)_0 C_1 H_1 + \left(\frac{\partial^2 F}{\partial x \partial x'}\right)_0 C_1 C_1' + \left(\frac{\partial^2 F}{\partial x \partial y'}\right)_0 C_1 H_1' + \left(\frac{\partial^2 F}{\partial x^1 \partial y}\right)_0 C_1' H_1 + \left(\frac{\partial^2 F}{\partial x^1 \partial y^1}\right)_0 C_1' H_1' + \left(\frac{\partial^2 F}{\partial y \partial y'}\right)_0 H_1 H_1' + \left(\frac{\partial^2 F}{\partial x' \partial \mu}\right)_0 C_1' + \left(\frac{\partial^2 F}{\partial y \partial \mu}\right)_0 H_1 + \left(\frac{\partial^2 F}{\partial x \partial \mu}\right)_0 C_1 + \left(\frac{\partial^2 F}{\partial y' \partial \mu}\right) H_1' + \left(\frac{\partial F}{\partial x}\right)_0 C_2 + \left(\frac{\partial F}{\partial x'}\right)_0 C_2' + \left(\frac{\partial F}{\partial y}\right)_0 H_2 + \left(\frac{\partial F}{\partial y'}\right)_0 H_2'$$

Analogous formulas exist for ϕ_n. The subscript 0 at the parentheses indicates that the x, x', y, y' and μ are replaced by x_0, x_0', y_0, y_0' and 0 in the derivatives.

Having solved the system (1.8) and taking into account the relations

$$-\frac{c}{k} = p_k \frac{d-k^2}{k}, \quad -\frac{b}{k} p_k = \frac{a-k^2}{k}, \quad -\frac{c}{m} = p_m \frac{d-m^2}{m}, \quad -\frac{b}{m} p_m = \frac{a-m^2}{m}$$

one can express the functions $C_n(t)$ and $H_n(t)$ in the form

$$C_n(t) = C_n^{(k)}(t) + C_n^{(m)}(t), \qquad H_n(t) = p_k C_n^{(k)}(t) + p_m C_n^{(m)}(t)$$

where

$$C_n^{(k)}(t) = \frac{1}{m^2-k^2}\left[\frac{d-k^2}{k}\int_0^t F_n(\tau)\sin k(t-\tau)d\tau - \frac{b}{k}\int_0^t \Phi_n(\tau)\sin k(t-\tau)\,d\tau\right]$$

$$C_n^{(m)}(t) = \frac{1}{m^2-k^2}\left[\frac{d-m^2}{m}\int_0^t F_n(\tau)\sin m(t-\tau)\,d\tau - \frac{b}{m}\int_0^t \Phi_n(\tau)\sin m(t-\tau)d\tau\right]$$

It is easy to show that the particular periodic solutions $f^\circ(t)$ and $\phi^\circ(t)$ can be represented as the sum of two terms $f_k(t)$ and $f_m(t)$ so that

$$f^\circ(t) = f_k(t) + f_m(t), \qquad \varphi^\circ(k) = p_k f_k(t) + p_m f_m(t)$$

In the case of nonautonomous systems, as well as in the case of autonomous ones, one can show that differentiation of $C_n^{(k)}$ and $C_n^{(m)}$ with respect to β_1, β_2, β_3 and β_4 is replaced by differentiation with respect to A_0, B_0, E_0 and D_0, respectively.

The final solution $x(t, \beta_1, \beta_2, \beta_3, \beta_4, \mu)$ and $y(t, \beta_1, \beta_2, \beta_3, \beta_4, \mu)$ of the system (1.1) is expressed in the form

$$x(t) = x_k(t) + x_m(t) + f_k(t) + f_m(t)$$
$$y(t) = p_k[x_k(t) + f_k(t)] + p_m[x_m(t) + f_m(t)] \tag{1.9}$$

where

$$x_k(t) = (A_0+\beta_1)\cos kt + \frac{B_0+\beta_2}{k}\sin kt + \sum_{n=1}^{\infty}\left[C_n^{(k)} + \frac{\partial C_n^{(k)}}{\partial A_0}\beta_1 + \right.$$
$$\left. + \frac{\partial C_n^{(k)}}{\partial B_0}\beta_2 + \frac{\partial C_n^{(k)}}{\partial E_0}\beta_3 + \frac{\partial C_n^{(k)}}{\partial D_0}\beta_4 + \frac{1}{2}\frac{\partial^2 C_n^{(k)}}{\partial A_0^2}\beta_1^2 + \ldots\right]\mu^n$$

$$x_m(t) = (E_0+\beta_3)\cos mt + \frac{D_0+\beta_4}{m}\sin mt + \tag{1.10}$$
$$+ \sum_{n=1}^{\infty}\left[C_n^{(m)} + \frac{\partial C_n^{(m)}}{\partial A_0}\beta_1 + \frac{\partial C_n^{(m)}}{\partial B_0}\beta_2 + \ldots\right]\mu^n$$

Hence, for the construction of periodic solutions of a nonautonomous quasi-linear system with two degrees of freedom it is sufficient to construct the functions x_k, x_m, f_k and f_m which enter into the x-coordinate.

The variable y is constructed by multiplying x_k, x_m, f_k and f_m by constants and adding the result, just as in the linear system.

In case (b), Formulas (1.9) take on the form

$$x(t) = x_k(t) + f_k(t), \qquad y(t) = p_k[x_k(t) + f_k(t)]$$

In case (c), Equations (1.2) can be factored and the solution can be written in the form

$$x(t) = (A_0 + \beta_1)\cos kt + \frac{B_0 + \beta_2}{k}\sin kt +$$

$$+ \sum_{n=1}^{\infty}\left[C_n + \frac{\partial C_n}{\partial A_0}\beta_1 + \frac{\partial C_n}{\partial B_0}\beta_2 + \ldots\right]\mu^n + f^{\circ}(t)$$

$$y(t) = (E_0 + \beta_3)\cos kt + \frac{D_0 + \beta_4}{k}\sin kt +$$

$$+ \sum_{n=1}^{\infty}\left[H_n + \frac{\partial H_n}{\partial A_0}\beta_1 + \frac{\partial H_n}{\partial B_0}\beta_2 + \ldots\right]\mu^n + \varphi^{\circ}(t)$$

Here, the coordinates x and y are not interconnected (which is also the case in linear systems).

For the construction of these solutions one has to know how to compute the coefficients $C_n^{(k)}(t)$ and $C_n^{(m)}(t)$ of μ^n. The remaining coefficients of the series (1.10) are found by successive differentiations of $c_n^{(k)}$ and $C_n^{(m)}$ with respect to A_0, B_0, E_0 and D_0.

2. Taking into account the conditions (1.6) we can write down the conditions for the periodic functions in the following form:

$$\begin{aligned} x(2\pi, \beta_1, \beta_2, \beta_3, \beta_4, \mu) &= f^{(0)}(0) + A_0 + \beta_1 + E_0 + \beta_3 \\ x'(2\pi, \beta_1, \beta_2, \beta_3, \beta_4, \mu) &= f^{(0)\prime}(0) + B_0 + \beta_2 + D_0 + \beta_4 \\ y(2\pi, \beta_1, \beta_2, \beta_3, \beta_1, \mu) &= \varphi^{(0)}(0) + (A_0 + \beta_1)p_k + (E_0 + \beta_3)p_m \\ y'(2\pi, \beta_1, \beta_2, \beta_3, \beta_4, \mu) &= \varphi^{(0)\prime}(0) + (B_0 + \beta_2)p_k + (D_0 + \beta_4)p_m \end{aligned} \tag{2.1}$$

Substituting into the left-hand sides of these equations the expressions for x, x', y and y' from (1.10), we obtain

$$\sum_{n=1}^{\infty}\left[C_n^{(k)}(2\pi) + \frac{\partial C_n^{(k)}}{\partial A_0}\beta_1 + \frac{\partial C_n^{(k)}}{\partial B_0}\beta_2 + \ldots\right]\mu^n +$$

$$+ \sum_{n=1}^{\infty}\left[C_n^{(m)}(2\pi) + \frac{\partial C_n^{(m)}}{\partial A_0}\beta_1 + \frac{\partial C_n^{(m)}}{\partial B_0}\beta_2 + \ldots\right]\mu^n = 0 \tag{2.2}$$

$$p_k \sum_{n=1}^{\infty} \left[C_n^{(k)}(2\pi) + \frac{\partial C_n^{(k)}}{\partial A_0} \beta_1 + \dots \right] \mu^n +$$
$$+ p_m \sum_{n=1}^{\infty} \left[C_n^{(m)}(2\pi) + \frac{\partial C_n^{(m)}}{\partial A_0} + \dots \right] \mu^n = 0$$

and two more formulas in which $C_n^{(k)}$ and $C_n^{(m)}$ are replaced by $C_n^{(k)'}$ and $C_n^{(m)'}$. The functions $C_n^{(k)}$, $C_n^{(k)'}$, $C_n^{(m)}$, $C_n^{(m)'}$ and their derivatives with respect to A_0, B_0, E_0 and D_0 are here evaluated at the point $t = 2\pi$, $\beta_i = \mu = 0$. Since $p_k - p_m \neq 0$, we have in place of (2.2) the next set of equations:

$$\begin{aligned} &\sum_{n=1}^{\infty} \left[C_n^{(k)}(2\pi) + \frac{\partial C_n^{(k)}}{\partial A_0} \beta_1 + \dots \right] \mu^n = 0, \\ &\sum_{n=1}^{\infty} \left[C_n^{(k)'}(2\pi) + \frac{\partial C_n^{(k)'}}{\partial A_0} \beta_1 + \dots \right] \mu^n = 0 \\ &\sum_{n=1}^{\infty} \left[C_n^{(m)}(2\pi) + \frac{\partial C_n^{(m)}}{\partial A_0} \beta_1 + \dots \right] \mu^n = 0 \\ &\sum_{n=1}^{\infty} \left[C_n^{(m)'}(2\pi) + \frac{\partial C_n^{(m)'}}{\partial A_0} \beta_1 + \dots \right] \mu^n = 0 \end{aligned} \qquad (2.3)$$

Let us assume that β_1, β_2, β_3 and β_4 can be expanded into series of the form

$$\beta_1 = \sum_{n=1}^{\infty} A_n \mu^n, \qquad \beta_2 = \sum_{n=1}^{\infty} B_n \mu^n, \qquad \beta_3 = \sum_{n=1}^{\infty} E_n \mu^n, \qquad \beta_4 = \sum_{n=1}^{\infty} D_n \mu^n \qquad (2.4)$$

Let us substitute β_i into the left-hand sides of Equations (2.3), and equate to zero the coefficients of the series. The terms which are independent of μ yield the results:

$$C_1^{(k)}(2\pi) = 0, \qquad C_1^{(m)}(2\pi) = 0, \qquad C_1^{(k)'}(2\pi) = 0, \qquad C_1^{(m)'}(2\pi) = 0 \qquad (2.5)$$

The coefficients of the first power of μ lead to the equations

$$C_2^{(k)}(2\pi) + \frac{\partial C_1^{(k)}}{\partial A_0} A_1 + \frac{\partial C_1^{(k)}}{\partial B_0} B_1 + \frac{\partial C_1^{(k)}}{\partial E_0} E_1 + \frac{\partial C_1^{(k)}}{\partial D_0} D_1 = 0$$
$$C_2^{(m)}(2\pi) + \frac{\partial C_1^{(m)}}{\partial A_0} A_1 + \frac{\partial C_1^{(m)}}{\partial B_0} B_1 + \frac{\partial C_1^{(m)}}{\partial E_0} E_1 + \frac{\partial C_1^{(m)}}{\partial D_0} D_1 = 0$$

$$C_2^{(k)'}(2\pi) + \frac{\partial C_1^{(k)'}}{\partial A_0} A_1 + \frac{\partial C_1^{(k)'}}{\partial B_0} B_1 + \frac{\partial C_1^{(k)'}}{\partial E_0} E_1 + \frac{\partial C_1^{(k)'}}{\partial D_0} D_1 = 0$$
$$C_2^{(m)'}(2\pi) + \frac{\partial C_1^{(m)'}}{\partial A_0} A_1 + \frac{\partial C_1^{(m)'}}{\partial B_0} B_1 + \frac{\partial C_1^{(m)'}}{\partial E_0} E_1 + \frac{\partial C_1^{(m)'}}{\partial D_0} D_1 = 0 \qquad (2.6)$$

The coefficients of the second-degree terms in μ yield

$$C_3^{(k)}(2\pi) + \frac{\partial C_2^{(k)}}{\partial A_0} A_1 + \frac{\partial C_2^{(k)}}{\partial B_0} B_1 + \frac{\partial C_2^{(k)}}{\partial E_0} E_1 + \frac{\partial C_2^{(k)}}{\partial D_0} D_1 +$$
$$+ \frac{\partial C_1^{(k)}}{\partial A_0} A_2 + \frac{\partial C_1^{(k)}}{\partial B_0} B_2 + \frac{\partial C_1^{(k)}}{\partial E_0} E_2 + \frac{\partial C_1^{(k)}}{\partial D_0} D_2 + \frac{1}{2}\frac{\partial^2 C_1^{(k)}}{\partial A_0^2} A_1^2 + \frac{1}{2}\frac{\partial^2 C_1^{(k)}}{\partial B_0^2} B_1^2 +$$
$$+ \frac{1}{2}\frac{\partial^2 C_1^{(k)}}{\partial E_0^2} E_1^2 + \frac{1}{2}\frac{\partial^2 C_1^{(k)}}{\partial D_0^2} D_1^2 + \frac{\partial^2 C_1^{(k)}}{\partial A_0 \partial B_0} A_1 B_1 + \frac{\partial^2 C_1^{(k)}}{\partial A_0 \partial E_0} A_1 E_1 +$$
$$+ \frac{\partial^2 C_1^{(k)}}{\partial A_0 \partial D_0} A_1 D_1 + \frac{\partial^2 C_1^{(k)}}{\partial B_0 \partial E_0} B_1 E_1 + \frac{\partial^2 C_1^{k}}{\partial B_0 \partial D_0} B_1 D_1 + \frac{\partial^2 C_1^{(k)}}{\partial E_0 \partial D_0} E_1 D_1 = 0 \qquad (2.7)$$

and three more equations in which the $C_i^{(k)}$ are successively replaced by $C_i^{(m)}$, $C_i^{(k)'}$ and $C_i^{(m)'}$. The system of equations (2.5) determines the constants A_0, B_0, B_0 and D_0 when these equations have simple roots, i.e. when the Jacobian

$$\Delta_1 = \frac{\partial\,(C_1^{(k)},\ C_1^{(m)},\ C_1^{(k)'},\ C_1^{(m)'})}{\partial\,(A_0,\ B_0,\ E_0,\ D_0)} \neq 0 \qquad (2.8)$$

In this case we determine A_1, B_1, E_1 and D_1 by means of the linear system (2.6), and we find A_2, B_2, E_2 and C_2 from (2.7), and so on. All these equations are linear in A_n, B_n, E_n and D_n, and all have the same determinant Δ_1.

In case of repeated roots of the system of equations (2.5), the determinant $\Delta_1 = 0$. If there is to be a periodic solution with finite amplitude of the system (1.1), it is necessary that an auxiliary condition be satisfied: the rank of the fundamental matrix of the linear system (2.6) and that of the augmented matrix (obtained by attaching a column of the free terms) must be the same. If this condition is satisfied then there can occur a bifurcation of the solution of the generating equations. If this condition is not satisfied, then the system of equations (2.6) can lead to infinite values for the coefficients A_1, B_1, E_1 and D_1. In this case the periodic solution of Equation (1.1) cannot be found by this method.

In all those cases when there exists a periodic solution of the system (1.1), this solution can be represented in the form of a power series in μ:

$$x(t) = x_0(t) + \mu x_1(t) + \mu^2 x_2(t) + \ldots, \quad y(t) = y_0(t) + \mu y_1(t) + \mu^2 y_2(t) + \ldots \qquad (2.9)$$

The generating solution is given by (1.3) or by (1.4) and (1.5). Here

$$x_1(t) = A_1 \cos kt + \frac{B_1}{k} \sin kt + E_1 \cos mt + \frac{D_1}{m} \sin mt + C_1^{(k)}(t) + C_1^{(m)}(t)$$

$$x_2(t) = A_2 \cos kt + \frac{B_2}{k} \sin kt + E_2 \cos mt + \frac{D_2}{m} \sin mt +$$
$$+ A_1 \left[\frac{\partial C_1^{(k)}}{\partial A_0} + \frac{\partial C_1^{(m)}}{\partial A_0} \right] + B_1 \left[\frac{\partial C_1^{(k)}}{\partial B_0} + \frac{\partial C_1^{(m)}}{\partial B_0} \right] +$$
$$+ E_1 \left[\frac{\partial C_1^{(k)}}{\partial E_0} + \frac{\partial C_1^{(m)}}{\partial E_0} \right] + D_1 \left[\frac{\partial C_1^{(k)}}{\partial D_0} + \frac{\partial C_1^{(m)}}{\partial D_0} \right] + C_2^{(k)}(t) + C_2^{(m)}(t)$$

$$y_1(t) = p_k \left[A_1 \cos kt + \frac{B_1}{k} \sin kt \right] + p_m \left[E_1 \cos mt + \frac{D_1}{m} \sin mt \right] +$$
$$+ p_k C_1^{(k)}(t) + p_m C_1^{(m)}(t)$$

$$y_2(t) = p_k \left[A_2 \cos kt + \frac{B_2}{k} \sin kt \right] + p_m \left[E_2 \cos mt + \frac{D_2}{m} \sin mt \right] + p_k C_2^{(k)}(t) +$$
$$+ A_1 \left[p_k \frac{\partial C_1^{(k)}}{\partial A_0} + p_m \frac{\partial C_1^{(m)}}{\partial A_0} \right] + B_1 \left[p_k \frac{\partial C_1^{(k)}}{\partial B_0} + p_m \frac{\partial C_1^{(m)}}{\partial B_0} \right] + p_m C_2^{(m)}(t) +$$
$$+ E_1 \left[p_k \frac{\partial C_1^{(k)}}{\partial E_0} + p_m \frac{\partial C_1^{(m)}}{\partial E_0} \right] + D_1 \left[p_k \frac{\partial C_1^{(k)}}{\partial D_0} + p_m \frac{\partial C_1^{m}}{\partial D_0} \right] \quad \text{etc.}$$

We have analysed above the case (a). For the cases (b) and (c) one obtains different forms. These results can be extended to systems with n degrees of freedom.

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Translated by H.P.T.